An Adaptation of the Classical CAPM to Insurance: The Weighted Insurance Pricing Model

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Abstract. We present and discuss an insurance version of the classical Capital Asset Pricing Model that offers economic pricing and risk capital allocation rules for a large class of risks, including those that are non-symmetric and heavy tailed. A number of illustrative examples are given, and convenient computational formulas suggested.

Key words and phrases: capital asset pricing model, weighted insurance pricing model, Gini-type insurance pricing model, beta, Gini correlation.

1. Introduction

The Capital Asset Pricing Model (CAPM) has profoundly influenced Finance and Insurance, with numerous articles and books written on the topic by academics and practitioners (e.g., Levy, 2011; and references therein). In this paper we aim at modifying the classical CAPM to accommodate some of the 'peculiarities' of insurance risks, in particular their positivity, skewness, and heavy tails.

We start with the obvious. Namely, the classical CAPM links the expected riskiness of portfolio constituents with the overall portfolio riskiness. Specifically, expressed in its classical form, the CAPM equation is

$$\mathbf{E}[R_i] = r_f + \beta_i\big(\mathbf{E}[R_m] - r_f\big), \tag{1.1}$$

where $\mathbf{E}[R_i]$ is the expected return on asset i, $\mathbf{E}[R_m]$ is the expected market rate of return, r_f is the risk-free rate of return, and β_i is the proportionality coefficient, widely known as 'beta' and given by the equation

$$\beta_i = \frac{\mathbf{Cov}[R_i, R_m]}{\mathbf{Cov}[R_m, R_m]}. \tag{1.2}$$

At this initial point of our discussion, it is instructive to recall the classical linear regression equation, which, under the assumption of bivariate normality on the pair (R_i, R_m), says that the conditional expectation $\mathbf{E}[R_i \mid R_m = s]$, as a function of s, is the straight line $a + bs$ with the slope $b = \beta_i$ and the intercept $a = \mathbf{E}[R_i] - \beta_i \mathbf{E}[R_m]$. This representation and its similarity to CAPM equation (1.1) explain the central role of the multivariate normal distribution in the CAPM literature.

Of course, departures from the normality assumption (e.g., Owen and Rabinovitch, 1983, for elliptical distributions) have been established and extensively discussed (Levy, 2011; and references therein). Indeed, risks generally deviate from symmetry and are often heavy tailed. In addition, insurance risks are as a rule positively valued (e.g., Klugman et al., 2008). Due to these and other reasons, when applying the CAPM equation to price insurance products or allocate capital to individual risks, we inevitably find ourselves in a position of doubt. In this paper, therefore, we propose to tweak the classical CAPM so that it would mitigate, if not resolve, the aforementioned issues. We illustrate the underlying idea in the next section using the so-called modified-variance risk measure and the corresponding risk capital allocation.

Throughout the rest of the paper, we use $X_1, X_2, \dots, X_d$ to denote real-valued (i.e., not necessarily positive) risk random variables whose aggregate riskiness is expressed by a random variable S (e.g., $S = X_1 + X_2 + \cdots + X_d$).

2. An illuminating example and our general aim

We start with an example illustrating that departure from normality is not difficult to achieve. To make our initial arguments as simple as possible, we work with the modified-variance risk measure (Heilmann, 1989)

$$\begin{aligned} \mathrm{mv}[S] &= \mathbf{E}[S] + \frac{1}{\mathbf{E}[S]} \mathbf{Var}[S] \\ &= \frac{\mathbf{E}[S^2]}{\mathbf{E}[S]} \end{aligned} \tag{2.1}$$

and, for $i \in \{1, \dots, d\}$, the corresponding risk capital allocation rule

$$\mathrm{MV}[X_i \mid S] = \frac{\mathbf{E}[X_i S]}{\mathbf{E}[S]}. \tag{2.2}$$

According to our view of the CAPM, we want to express $\mathrm{MV}[X_i \mid S]$, which measures the riskiness of X_i within the collection of risks, in terms of $\mathrm{mv}[S]$, which measures the aggregate riskiness. We achieve this goal with the help of simple algebra and, most importantly, without imposing any distributional constraints on the pair (X_i, S). Namely, we have the equations

$$
\begin{aligned}
\mathrm{MV}[X_i \mid S] - \mathbf{E}[X_i] &= \frac{\mathbf{E}[X_i S]}{\mathbf{E}[S]} - \mathbf{E}[X_i] \\
&= \frac{\mathbf{Cov}[X_i, S]}{\mathbf{E}[S]} \\
&= \frac{\mathbf{Cov}[X_i, S]}{\mathbf{Cov}[S, S]} \frac{\mathbf{Cov}[S, S]}{\mathbf{E}[S]} \\
&= \frac{\mathbf{Cov}[X_i, S]}{\mathbf{Cov}[S, S]} \frac{\mathbf{E}[S^2] - (\mathbf{E}[S])^2}{\mathbf{E}[S]} \\
&= \beta_i \big(\mathrm{mv}[S] - \mathbf{E}[S]\big), \qquad (2.3)
\end{aligned}
$$

where the 'beta'

$$
\beta_i = \frac{\mathbf{Cov}[X_i, S]}{\mathbf{Cov}[S, S]} \qquad (2.4)
$$

is of the same form as that given by equation (1.2).

Hence, in summary, in the modified-variance case, the insurance analogue of CAPM equation (1.1) is the equation

$$
\mathrm{MV}[X_i \mid S] = \mathbf{E}[X_i] + \beta_i \big(\mathrm{mv}[S] - \mathbf{E}[S]\big), \qquad (2.5)
$$

which holds for all pairs (X_i, S) for which $\mathrm{MV}[X_i \mid S]$ and $\mathrm{mv}[S]$ are well-defined and finite: no specific distribution on the risks has been imposed.

Despite the latter optimistic message, we still rely on the existence of finite second moments of the underlying random risks, but this is only due to our choice of the modified-variance risk measure and the capital allocation rule. To accommodate heavier-tailed risks, we therefore wish to depart from the above risk measure and the capital allocation rule, and for this we put forward a research program whose main idea hinges on the following modification of CAPM equation (1.1):

(1) replace the two risk-free rates of return r_f by the corresponding averages $\mathbf{E}[X_i]$ and $\mathbf{E}[S]$, frequently called net premiums in the actuarial literature;

(2) replace the expected market rate of return $\mathbf{E}[R_m]$ by a risk measure $\pi[S]$ of the aggregate risk S;

(3) replace the expected return $\mathbf{E}[R_i]$ on the asset i by a risk capital allocation rule $\Pi[X_i \mid S]$ due to the risk X_i;

(4) find, if possible, an appropriate proportionally coefficient β_i – which we keep calling 'beta' to maintain consistency with the already accepted terminology in the CAPM literature – that does not depend on any utility, weight, distortion, etc. 'subjective' function.

Hence, in various scenarios of practical interest, in what follows we aim at deriving the equation

$$\Pi[X_i \mid S] = \mathbf{E}[X_i] + \beta_i\big(\pi[S] - \mathbf{E}[S]\big), \tag{2.6}$$

which we generally call the insurance pricing model (IPM) equation. An important clarification is needed at this point in order to avoid a potential misunderstanding.

Namely, from the mathematical point of view, equation (2.6) always holds with $\beta_i = \beta_i(\Pi, \pi)$ defined by

$$\beta_i(\Pi, \pi) = \frac{\Pi[X_i \mid S] - \mathbf{E}[X_i]}{\pi[S] - \mathbf{E}[S]}, \tag{2.7}$$

whenever of course $\pi[S]$ is positively loaded, that is $\pi[S] > \mathbf{E}[S]$. The ratio of loadings $\beta_i(\Pi, \pi)$ may, in general, depend on 'subjective' functions (e.g., utility, weight, distortion, etc.) that define the risk capital allocation rule $\Pi[X_i \mid S]$ and the risk measure $\pi[S]$. But we say that equation (2.6) is the IPM equation only when $\beta_i(\Pi, \pi)$ does not depend on these functions. Hence, our proposed IPM hinges on the fact that under certain but quite general conditions, ratio (2.7) is independent of any subjective function, and it is only in this case that we call ratio (2.7) 'beta.'

In what follows, we discuss several versions of the IPM equation: the weighted insurance pricing model (WIPM) equation in Section 3, and the Gini-type weighted insurance pricing model (G-WIPM) equation in Section 4.

3. Weighted insurance pricing model

The weighted risk measure (Furman and Zitikis, 2008a, 2009), which is very general and allows us to accommodate virtually every risk irrespective of its tail-heaviness as long as we appropriately choose a weight function $w : (-\infty, \infty) \to [0, \infty)$, is defined by

$$\pi_w[S] = \frac{\mathbf{E}[Sw(S)]}{\mathbf{E}[w(S)]}. \tag{3.1}$$

The weight function w is usually assumed, or chosen, to be non-decreasing, which ensures, for example, non-negative loading of the risk measure. The corresponding weighted risk capital allocation rule is (Furman and Zitikis, 2008b)

$$\Pi_w[X_i \mid S] = \frac{\mathbf{E}[X_i w(S)]}{\mathbf{E}[w(S)]}. \tag{3.2}$$

For example, by choosing the weight functions $w : [0, \infty) \to [0, \infty)$ given by

$$w(s) = s^\lambda,$$

$$w(s) = e^{\lambda s},$$

$$w(s) = 1 - e^{-\lambda s},$$

$$w(s) = \mathbf{1}\{s > \lambda\},$$

where $\lambda > 0$ is a parameter, we reduce $\pi_w[S]$ to the size-biased, Esscher's, Kamps's, and excess-of-loss risk measures, and we in turn reduce $\Pi_w[X_i \mid S]$ to the corresponding risk capital allocation rules. A few other examples will follow later in this paper, but next we show how the IPM equation (which we call WIPM) arises in the case of the weighted risk measure and the corresponding capital allocation rule.

Using simple algebra and following the same route as in the modified-variance case, we obtain the equations

$$\begin{aligned}\Pi_w[X_i \mid S] - \mathbf{E}[X_i] &= \frac{\mathbf{E}[X_i w(S)]}{\mathbf{E}[w(S)]} - \mathbf{E}[X_i] \\ &= \frac{\mathbf{Cov}[X_i, w(S)]}{\mathbf{E}[w(S)]} \\ &= \frac{\mathbf{Cov}[X_i, w(S)]}{\mathbf{Cov}[S, w(S)]}\frac{\mathbf{Cov}[S, w(S)]}{\mathbf{E}[w(S)]} \\ &= \frac{\mathbf{Cov}[X_i, w(S)]}{\mathbf{Cov}[S, w(S)]}\frac{\mathbf{E}[Sw(S)] - \mathbf{E}[S]\mathbf{E}[w(S)]}{\mathbf{E}[w(S)]} \\ &= \beta_{i,w}\big(\pi_w[S] - \mathbf{E}[S]\big),\end{aligned} \tag{3.3}$$

where the ratio of loadings $\beta_{i,w}$ (we refrain from calling it 'beta' because it may, in general, depend on the 'subjective' weight function w) is given by the equation

$$\beta_{i,w} = \frac{\mathbf{Cov}[X_i, w(S)]}{\mathbf{Cov}[S, w(S)]}. \tag{3.4}$$

When, however, $\beta_{i,w}$ does not depend on w, that is, $\beta_{i,w} = \beta_i$ for some β_i, the above considerations give rise to the equation (cf. Furman and Zitikis, 2010)

$$\Pi_w[X_i \mid S] = \mathbf{E}[X_i] + \beta_i\big(\pi_w[S] - \mathbf{E}[S]\big), \tag{3.5}$$

which we call the WIPM equation, and which is our proposed insurance analogue of CAPM equation (1.1). Note that when $w(s) = s$, equation (3.5) reduces to equation (2.5), but this fact does not imply that β_i in equation (3.5) is the same as in equation (2.4), as we shall see in a moment. We next show the validity of WIPM equation (3.5) in two special cases.

Case 1: linear regression. Assume that the regression function

$$r_i(s) = \mathbf{E}[X_i \mid S = s] \tag{3.6}$$

is linear, that is,

$$r_i(s) = a + bs \tag{3.7}$$

for some constants a and b, called the intercept and the slope, respectively. Then

$$\begin{aligned}\beta_{i,w} &= \frac{\mathbf{Cov}[X_i, w(S)]}{\mathbf{Cov}[S, w(S)]} \\ &= \frac{\mathbf{Cov}[r_i(S), w(S)]}{\mathbf{Cov}[S, w(S)]} \\ &= \frac{\mathbf{Cov}[a + bS, w(S)]}{\mathbf{Cov}[S, w(S)]} \\ &= b\frac{\mathbf{Cov}[S, w(S)]}{\mathbf{Cov}[S, w(S)]} = b. \end{aligned} \tag{3.8}$$

Hence, the ratio of loadings $\beta_{i,w}$ is equal to the slope b, which is of course free of the weight function w and can thus be called 'beta.' In turn, WIPM equation (3.5) becomes

$$\Pi_w[X_i \mid S] = \mathbf{E}[X_i] + b\big(\pi_w[S] - \mathbf{E}[S]\big). \tag{3.9}$$

The regression function is linear in a number of popular multivariate risk models. We refer to Furman and Zitikis (2010), Su (2016), Su and Furman (2017), Furman and Zitikis (2016a,b) for examples, details, and further references. In particular, in these works we find expressions of the slope b in terms of distribution parameters, which can in turn be estimated using various techniques already available in the literature, such as the maximum likelihood method, the method of (trimmed) moments, and so on (e.g., Brazauskas et al., 2009; Kleefeld and Brazauskas, 2012; and references therein). We may also seek non-parametric estimators of b, which can be found in standard books on regression.

Case 2: linear regression and non-negative risks. Having mentioned non-negative risks, which are abundant in insurance and are called losses (e.g., Klugman et al., 2008), we now look at the case of non-negative risks X_i. Let the aggregate risk be the sum $S = X_1 + X_2 + \cdots + X_d$. Due to the obvious equations $\sum_i r_i(0) = \mathbf{E}[S \mid S = 0] = 0$ and the non-negativity of all the summands $r_i(0)$, we have $r_i(0) = 0$. This fact and linearity assumption (3.7) imply that the intercept of the regression line vanishes, that is, $a = 0$, and we thus in turn obtain the equation

$$b = \frac{\mathbf{E}[X_i]}{\mathbf{E}[S]} \tag{3.10}$$

because $\mathbf{E}[X_i] = \mathbf{E}[r_i(S)] = b\mathbf{E}[S]$. Consequently, WIPM equation (3.5) becomes

$$\begin{aligned}\Pi_w[X_i \mid S] &= \mathbf{E}[X_i] + \frac{\mathbf{E}[X_i]}{\mathbf{E}[S]}\big(\pi_w[S] - \mathbf{E}[S]\big) \\ &= \frac{\mathbf{E}[X_i]}{\mathbf{E}[S]}\pi_w[S]. \end{aligned} \tag{3.11}$$

Note that the 'beta' b given by equation (3.10) requires the existence of only the first moments of the risks X_i and S. This is in sharp contrast with the covariance-based betas that we encountered earlier in this paper.

4. Gini-type weighted insurance pricing model

There are many risk measures and risk capital allocation rules that are not covered in our previous discussion of $\pi_w[S]$ and $\Pi_w[X_i \mid S]$. The reason is that in a number of cases the weight function w acts not on the aggregate risk severity S but on its rank $F(S)$, where F is the cumulative distribution function (cdf) of S. Hence, we next turn $\pi_w[S]$ and $\Pi_w[X_i \mid S]$ into what we call the Gini-type weighted risk measure

$$\pi_{w,\mathrm{Gini}}[S] = \frac{\mathbf{E}[Sw(F(S))]}{\mathbf{E}[w(F(S))]} \tag{4.1}$$

and the corresponding Gini-type risk capital allocation rule

$$\Pi_{w,\mathrm{Gini}}[X_i \mid S] = \frac{\mathbf{E}[X_i w(F(S))]}{\mathbf{E}[w(F(S))]}. \tag{4.2}$$

To illustrate them, we use the weight functions $w : [0, 1] \to [0, \infty)$ given by

$$w(t) = p(1-t)^{p-1},$$
$$w(t) = g'(1-t),$$
$$w(t) = e^{pt},$$
$$w(t) = \mathbf{1}\{t > p\},$$

where $p > 0$ is a parameter, and $g : [0, 1] \to [0, 1]$ is a 'distortion' function (e.g., $g(t) = t^p$; Wang, 1995, 1996). The above examples of the weight function w give rise to, respectively, the proportional hazards, distortion, Aumann-Shapley, and conditional tail expectation risk measures, as well as to the corresponding risk capital allocation rules. In addition, the weight function

$$w(t) = w_0(t)\mathbf{1}\{t > p\}$$

with some 'underlying' weight function $w_0 : [0, 1] \to [0, \infty)$ leads to what we call the Gini-type conditional-tail weighted risk measure and the corresponding risk capital allocation rule (Furman and Zitikis, 2016a,b). For more extensive mathematical details on this topic, we refer to Furman et al (2017).

To derive the corresponding pricing model, we follow equations (3.3) with $w(F(S))$ instead of $w(S)$ and have

$$\Pi_{w,\mathrm{Gini}}[X_i \mid S] = \mathbf{E}[X_i] + \beta_{i,w,\mathrm{Gini}}\big(\pi_{w,\mathrm{Gini}}[S] - \mathbf{E}[S]\big), \tag{4.3}$$

where the ratio of loadings is

$$\beta_{i,w,\mathrm{Gini}} = \frac{\mathbf{Cov}[X_i, w(F(S))]}{\mathbf{Cov}[S, w(F(S))]}. \tag{4.4}$$

When the ratio of loadings $\beta_{i,w,\mathrm{Gini}}$ does not depend on w, in which case we denote it by $\beta_{i,\mathrm{Gini}}$, we arrive at the equation

$$\Pi_{w,\mathrm{Gini}}[X_i \mid S] = \mathbf{E}[X_i] + \beta_{i,\mathrm{Gini}}\big(\pi_{w,\mathrm{Gini}}[S] - \mathbf{E}[S]\big) \tag{4.5}$$

that we call the G-WIPM equation. A remark on $\beta_{i,w,\mathrm{Gini}}$ follows next, after which we discuss two cases when $\beta_{i,w,\mathrm{Gini}}$ does not depend on w.

Interestingly, the ratio of covariances on the right-hand side of equation (4.4) has already appeared in the literature. Namely, when $w(t) = t$, the ratio is known as the Gini correlation coefficient and is usually denoted by $\Gamma[X_i, S]$. Its origins can be traced back to the work of C. Gini a hundred years ago. The ratio of covariances under the weight function $w(t) = p(1-t)^{p-1}$ is usually denoted by $\Gamma_p[X_i, S]$ and called the extended Gini correlation coefficient. It appeared and was thoroughly investigated several decades ago in the works of S. Yitzhaki and E. Schechtman, and we refer to the recent monograph by Yitzhaki and Schechtman (2013) for details and references on the topic. In our CAS technical report (Furman and Zitikis, 2016a), we discuss the weighted Gini-type correlation coefficient in the same form as presented on the right-hand side of equation (4.4). We next discuss two cases when $\beta_{i,w,\mathrm{Gini}}$ does not depend on w.

Case 1: linear regression. A sufficient condition for the G-WIPM equation to hold is the linearity of the regression function $r_i(s)$, that is, when equation (3.7) holds. Indeed, in this case we can follow equations (3.8) with $w(F(S))$ instead of $w(S)$ and obtain that $\beta_{i,w,\mathrm{Gini}}$ is equal to the slope b of the regression line, which is of course independent of w and can thus be called 'beta.' In this case, analogously to WIPM equation (3.9), we have the following G-WIPM equation

$$\Pi_{w,\mathrm{Gini}}[X_i \mid S] = \mathbf{E}[X_i] + b\big(\pi_{w,\mathrm{Gini}}[S] - \mathbf{E}[S]\big). \tag{4.6}$$

Case 2: linear regression and non-negative risks. When in addition to linearity of the regression function we also deal with non-negative risks, the G-WIPM equation turns into

$$\begin{aligned}\Pi_{w,\mathrm{Gini}}[X_i \mid S] &= \mathbf{E}[X_i] + \frac{\mathbf{E}[X_i]}{\mathbf{E}[S]}\big(\pi_{w,\mathrm{Gini}}[S] - \mathbf{E}[S]\big) \\ &= \frac{\mathbf{E}[X_i]}{\mathbf{E}[S]}\pi_{w,\mathrm{Gini}}[S],\end{aligned} \tag{4.7}$$

which is an analogue of WIPM equation (3.11).

5. On the independence of loading ratios of w

In our explorations above, we have encountered three ratios of loadings: the most general $\beta_i(\Pi, \pi)$ in equation (2.7), the weighted ratio $\beta_{i,w}$ in equation (3.4), and the Gini-type weighted ratio $\beta_{i,w,\text{Gini}}$ in equation (4.4). As in the classical CAPM, neither of these ratios we want to depend on any 'subjective' function, such as the weight function w.

A parametric route for checking whether this is true or not has transpired in our considerations above. Namely, given data, we can start with a goodness-of-fit technique and choose an appropriate parametric distribution for (X_i, S). Then, mathematically, we would derive an expression for the regression function $r_i(s)$ and see whether it is linear or not (e.g., Furman and Zitikis, 2016a,b; and references therein).

The parametric approach, however, is usually quite time and energy consuming, given the mathematical complexities that naturally arise when calculating conditional densities and, in turn, the regression function $r_i(s)$. Hence, one would – at least initially – prefer a simpler non-parametric method (some kind of a 'rule of thumb') to determine whether the loading ratio could, or could not, be free of any 'subjective' function. We offer several thoughts on this issue that we think might be helpful in practice.

Ratio $\beta_{i,w}$: a computational formula. Perhaps the simplest way that we can think of for checking if the ratio $\beta_{i,w}$ might be independent of w would be to construct a non-parametric estimate of the ratio and then plug into it several specific weight functions w to see whether there would be any significant change in the obtained estimates. This is definitely a heuristic approach, but we believe it is fast and practically useful. Hence, suppose that we have n observed pairs $(x_{i,k}, s_k)$, $1 \le k \le n$. In this case,

$$\beta_{i,w} = \frac{\mathbf{E}[X_i w(S)] - \mathbf{E}[X_i]\mathbf{E}[w(S)]}{\mathbf{E}[S w(S)] - \mathbf{E}[S]\mathbf{E}[w(S)]} \approx \widehat{\beta}_{i,w}, \tag{5.1}$$

where

$$\widehat{\beta}_{i,w} = \frac{\sum_{k=1}^n x_{i,k} w(s_k) - \widehat{x}_i \sum_{k=1}^n w(s_k)}{\sum_{k=1}^n s_k w(s_k) - \widehat{s} \sum_{k=1}^n w(s_k)} \tag{5.2}$$

with

$$\widehat{x}_i = \frac{1}{n}\sum_{k=1}^n x_{i,k} \quad \text{and} \quad \widehat{s} = \frac{1}{n}\sum_{k=1}^n s_k. \tag{5.3}$$

The dependence of $\widehat{\beta}_{i,w}$ on w can now be explored numerically.

Ratio $\beta_{i,w}$: an alternative computational formula. There might be situations when in addition to realizations s_k, $1 \le k \le n$, there is also an estimate $\widehat{r}_i(s)$ of the regression function $r_i(s)$. In this case,

$$\beta_{i,w} = \frac{\mathbf{Cov}[r_i(S), w(S)]}{\mathbf{Cov}[S, w(S)]} \approx \widetilde{\beta}_{i,w}, \tag{5.4}$$

where

$$\widetilde{\beta}_{i,w} = \frac{\sum_{k=1}^{n} \widehat{r}_i(s_k)w(s_k) - \widetilde{x}_i \sum_{k=1}^{n} w(s_k)}{\sum_{k=1}^{n} s_k w(s_k) - \widehat{s} \sum_{k=1}^{n} w(s_k)} \tag{5.5}$$

with

$$\widetilde{x}_i = \frac{1}{n} \sum_{k=1}^{n} \widehat{r}_i(s_k). \tag{5.6}$$

Based on this, we can now explore the dependence of $\widetilde{\beta}_{i,w}$ on w numerically.

Ratio $\beta_{i,w,\mathrm{Gini}}$: a computational formula. Suppose that we have observed pairs $(x_{i,k}, s_k)$, $1 \le k \le n$. Using them, we estimate $\mathbf{E}[X_i]$ and $\mathbf{E}[S]$ by $\widehat{x}_i$ (or $\widetilde{x}_i$) and $\widehat{s}$, respectively, whose definitions are given above. Then

$$\beta_{i,w,\mathrm{Gini}} = \frac{\mathbf{E}[X_i w(F(S))] - \mathbf{E}[X_i] \int_0^1 w(u)\mathrm{d}u}{\mathbf{E}[S w(F(S))] - \mathbf{E}[S] \int_0^1 w(u)\mathrm{d}u} \approx \widehat{\beta}_{i,w,\mathrm{Gini}} \tag{5.7}$$

with

$$\widehat{\beta}_{i,w,\mathrm{Gini}} = \frac{\sum_{k=1}^{n} x_{i,k,n}^{*} w(k/n) - \left(\sum_{k=1}^{n} x_{i,k}\right) \int_0^1 w(u)\mathrm{d}u}{\sum_{k=1}^{n} s_{k:n} w(k/n) - \left(\sum_{k=1}^{n} s_k\right) \int_0^1 w(u)\mathrm{d}u}, \tag{5.8}$$

where $s_{1:n} \le s_{2:n} \le \cdots \le s_{n:n}$ are ordered s_k, $1 \le k \le n$, and where $x_{i,k,n}^{*}$, $1 \le k \le n$, are the first coordinates of the pairs $(x_{i,k}, s_k)$, $1 \le k \le n$, ordered according to the ascending second coordinates. In the statistical literature, $x_{i,k,n}^{*}$, $1 \le k \le n$, are called the induced (by s_k, $1 \le k \le n$) order statistics of $x_{i,k}$, $1 \le k \le n$. We can now explore the dependence of $\widehat{\beta}_{i,w,\mathrm{Gini}}$ on w numerically.

Ratio $\beta_{i,w,\mathrm{Gini}}$: an alternative computational formula. We may also proceed by connecting $\beta_{i,w,\mathrm{Gini}}$ with so-called L-estimates. To somewhat simplify the presentation, we assume that the cdf F of S is a continuous function, in which case $F(S)$ is a uniform on $[0, 1]$ random variable. Using the classical notation F^{-1} for the inverse (i.e., quantile) function of F, and with the regression function $r_i(s)$ defined by equation (3.6), we have the equations

$$\begin{aligned}
\beta_{i,w,\mathrm{Gini}} &= \frac{\mathbf{E}[X_i w(F(S))] - \mathbf{E}[X_i] \int_0^1 w(u)\mathrm{d}u}{\mathbf{E}[S w(F(S))] - \mathbf{E}[S] \int_0^1 w(u)\mathrm{d}u} \\
&= \frac{\mathbf{E}[r_i(S) w(F(S))] - \mathbf{E}[X_i] \int_0^1 w(u)\mathrm{d}u}{\mathbf{E}[S w(F(S))] - \mathbf{E}[S] \int_0^1 w(u)\mathrm{d}u} \\
&= \frac{\int_0^1 r_i(F^{-1}(u)) w(u)\mathrm{d}u - \mathbf{E}[X_i] \int_0^1 w(u)\mathrm{d}u}{\int_0^1 F^{-1}(u) w(u)\mathrm{d}u - \mathbf{E}[S] \int_0^1 w(u)\mathrm{d}u}.
\end{aligned} \tag{5.9}$$

Given observed pairs $(x_{i,k}, s_k)$, $1 \le k \le n$, we estimate $\mathbf{E}[X_i]$ by $\widehat{x}_i$ (or $\widetilde{x}_i$) and $\mathbf{E}[S]$ by $\widehat{s}$. The integral

$$L_w = \int_0^1 F^{-1}(u) w(u)\mathrm{d}u$$

in the denominator on the right-hand side of equation (5.9) is known in the statistical literature as L-functional, and its estimate is

$$\widehat{L}_w = \sum_{k=1}^{n} s_{k:n} \int_{(k-1)/n}^{k/n} w(u)\mathrm{d}u,$$

where $s_{1:n} \leq s_{2:n} \leq \cdots \leq s_{n:n}$ are ordered s_k, $1 \leq k \leq n$. As to the integral

$$L_w(r_i) = \int_0^1 r_i(F^{-1}(u))w(u)\mathrm{d}u$$

in the numerator on the right-hand side of equation (5.9), we have the estimate

$$\widehat{L}_w(\widehat{r}_i) = \sum_{k=1}^{n} \widehat{r}_i(s_{k:n}) \int_{(k-1)/n}^{k/n} w(u)\mathrm{d}u,$$

where $\widehat{r}_i(s)$ is an estimate of the regression function $r_i(s)$. In summary, we have

$$\widetilde{\beta}_{i,w,\mathrm{Gini}} = \frac{\widehat{L}_w(\widehat{r}_i) - \widehat{x}_i \int_0^1 w(u)\mathrm{d}u}{\widehat{L}_w - \widehat{s} \int_0^1 w(u)\mathrm{d}u}. \tag{5.10}$$

We may of course use $\widetilde{x}_i$ given by equation (5.6) instead of $\widehat{x}_i$ on the right-hand side of equation (5.10). We can now explore the dependence of $\widetilde{\beta}_{i,w,\mathrm{Gini}}$ on w numerically.

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